Lecture 3
Foundation of Data Flow Analysis

I Semi-lattice (set of values, meet operator)
II Transfer functions
III Correctness, precision and convergence
IV Meaning of Data Flow Solution

Reading: Chapter 9.3

I. Purpose of a Framework

- **Purpose 1**
  - Prove properties of entire family of problems once and for all
    - Will the program converge?
    - What does the solution to the set of equations mean?
- **Purpose 2:**
  - Aid in software engineering: re-use code

The Data-Flow Framework

- **Data-flow problems (F, V, $\wedge$) are defined by**
  - A semi-lattice
    - domain of values V
    - meet operator $\wedge$: V × V → V
  - A family of transfer functions F: V → V

Semi-lattice: Structure of the Domain of Values

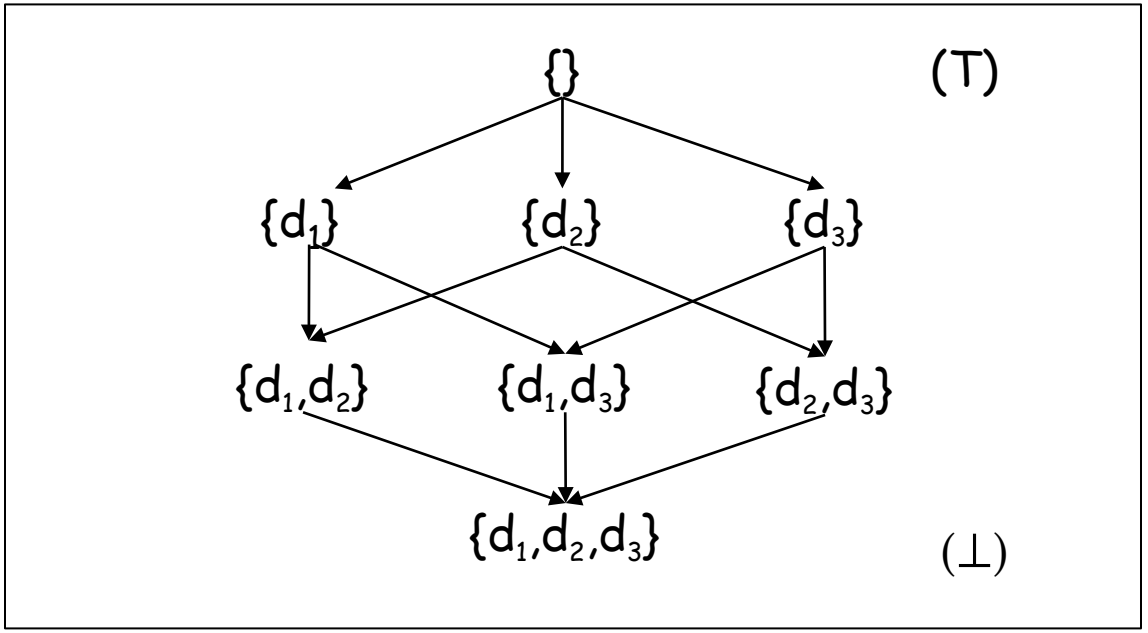
- **A semi-lattice S = <a set of values V, a meet operator ∧>**

- **Properties of the meet operator**
  - idempotent: $x \wedge x = x$
  - commutative: $x \wedge y = y \wedge x$
  - associative: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

- **Examples of meet operators ?**
- **Non-examples ?**

Example of a Semi-Lattice Diagram

- $(V, \wedge)$: $V = \{x \mid \text{such that } x \subseteq \{d_1, d_2, d_3\}\}$, $\wedge = \cup$



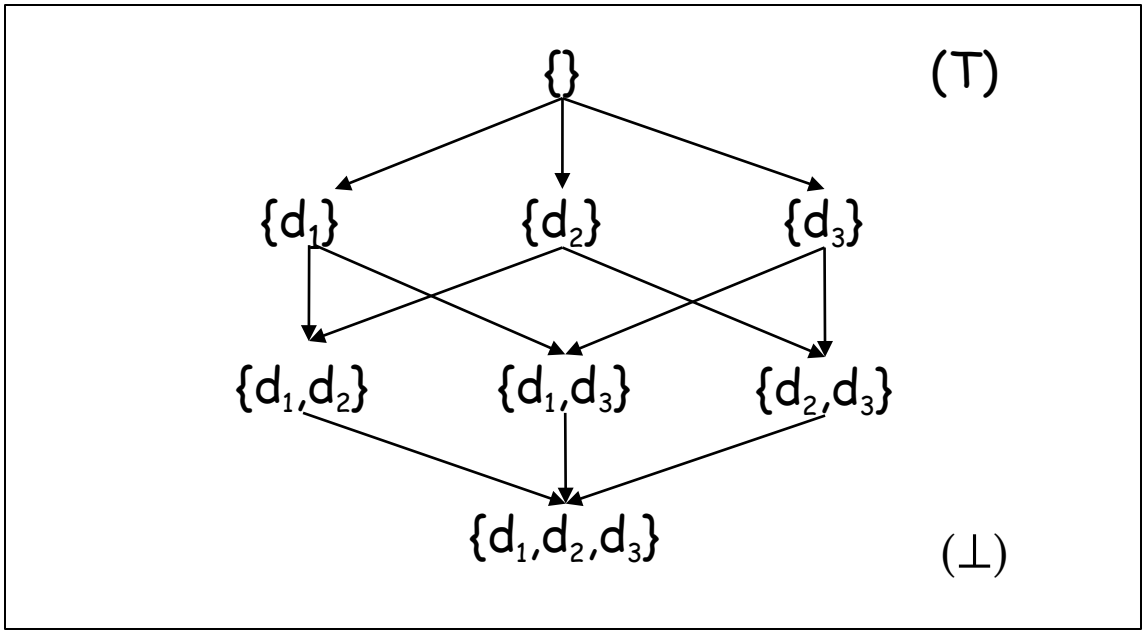
- $x \wedge y$ = first common descendant of x & y

important

- A meet semi-lattice is bounded if there exists a top element T, such that $x \wedge T = x$ for all x.
- A bottom element $\perp$ exists, if $x \wedge \perp = \perp$ for all x.

Meet Semi-Lattices vs Partially Ordered Sets

- A **meet-semilattice** is a partially ordered set which has a meet (or greatest lower bound) for any nonempty finite subset.



- Greatest lower bound: $x \wedge y$ = First common descendant of x & y
- Largest: top element T, if $x \wedge T = x$ for all x.
- Smallest: bottom element ⊥, if $x \wedge \bot = \bot$ for all x.

A Meet Operator Defines a Partial Order

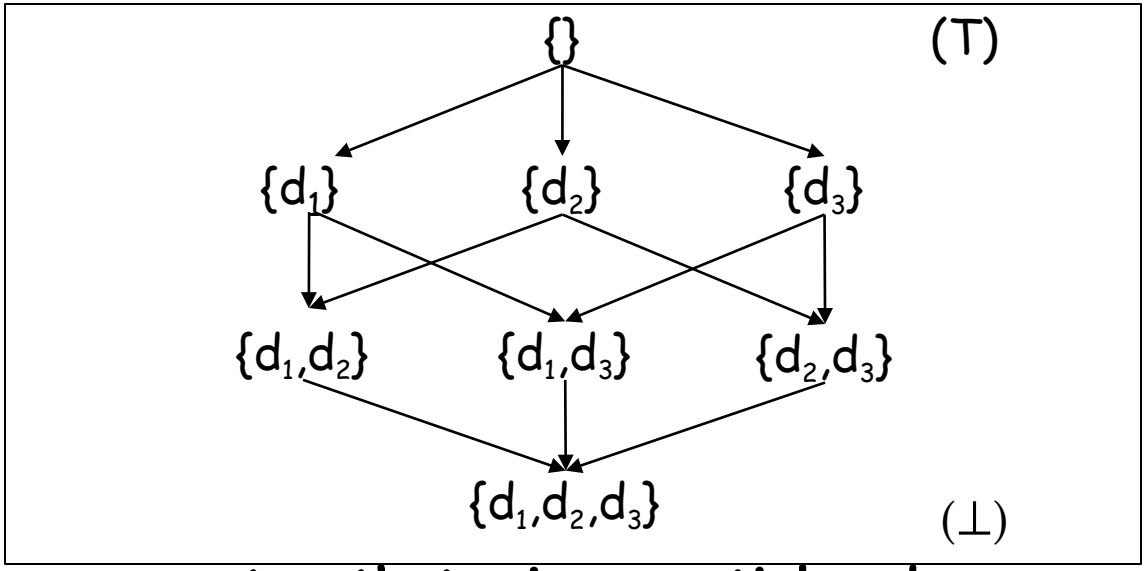
- **Partial order of a meet semi-lattice**

  $\leq$: $x \leq y$ if and only if $x \wedge y = x$

path: y ↓ x $\equiv$ $(x \wedge y = x)$ $\equiv$ $(x \leq y)$

- Meet operator: U

  Partial order $\leq$:



- **Properties of meet operator guarantee that $\leq$ is a partial order**
  - Reflexive: $x \leq x$
  - Antisymmetric: if $x \leq y$ and $y \leq x$ then $x = y$
  - Transitive: if $x \leq y$ and $y \leq z$ then $x \leq z$

Drawing a Semi-Lattice Diagram

- $(x < y) \equiv (x \leq y) \wedge (x \neq y)$

- ***A semi-lattice diagram:***
  - Set of nodes: set of values
  - Set of edges $\{(y, x)$: $x < y$ and $\neg \exists z$ s.t. $(x < z) \wedge (z < y)\}$

Summary

Three ways to define a semi-lattice:

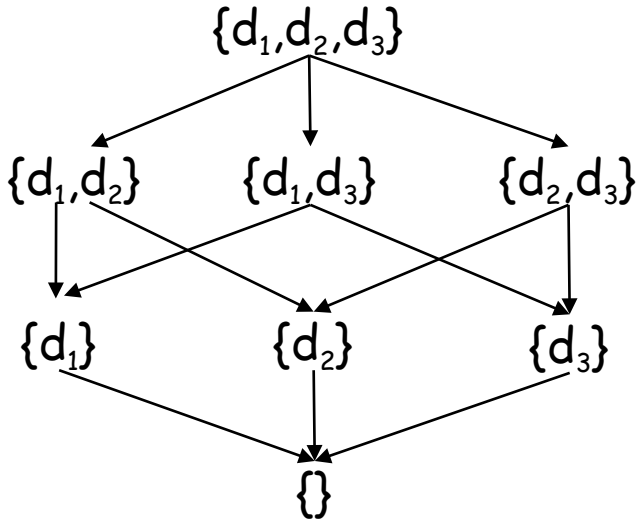
- Set of values + meet operator
  - idempotent: $x \wedge x = x$
  - commutative: $x \wedge y = y \wedge x$
  - associative: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

- Set of values
  + partial order with a greatest lower bound for any nonempty subset
  - Reflexive: $x \leq x$
  - Antisymmetric: if $x \leq y$ and $y \leq x$ then $x = y$
  - Transitive: if $x \leq y$ and $y \leq z$ then $x \leq z$

- A semi-lattice diagram

Another Example

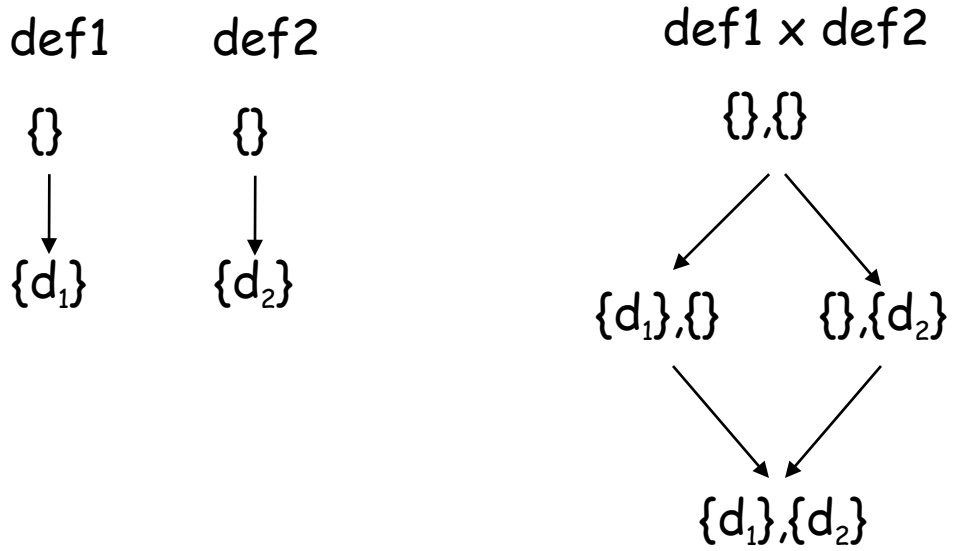
- Semi-lattice
  - V = {x | such that $x \subseteq \{d_1, d_2, d_3\}$}
  - $\wedge = \cap$



  - $\leq$ is

One Element at a Time

- **A semi-lattice for data flow problems can get quite large: 2^n elements for n var/definition**
- **A useful technique:**
  - define semi-lattice for 1 element
  - product of semi-lattices for all elements
- **Example**: Union of definitions
  - For each element



  - $\langle x_1, x_2 \rangle \leq \langle y_1, y_2 \rangle$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$

Descending Chain

- **Definition**
  - The **height** of a lattice is the largest number of > relations that will fit in a descending chain.

$$x_0 > x_1 > \ldots$$

- **Height of values in reaching definitions?**

- **Important property: finite descending chains**

II. Transfer Functions

- **A family of transfer functions** F
- **Basic Properties** $f : V \rightarrow V$
  - Has an identity function
    - $\exists f$ such that $f(x) = x$, for all x.
  - Closed under composition
    - if $f_1, f_2 \in F$, $f_1 \bullet f_2 \in F$

Monotonicity: 2 Equivalent Definitions

- A framework $(F, V, \wedge)$ is monotone iff
  - $x \leq y$ implies $f(x) \leq f(y)$

- Equivalently,
  a framework $(F, V, \wedge)$ is monotone iff
  - $f(x \wedge y) \leq f(x) \wedge f(y)$,
  - meet inputs, then apply f

    $\leq$

    apply f individually to inputs, then meet results

Example

- **Reaching definitions: $f(x) = Gen \cup (x - Kill)$, $\wedge = \cup$**
  - Definition 1:
    - Let $x_1 \leq x_2$,

      $f(x_1)$: Gen U $(x_1$ - Kill)

      $f(x_2)$: Gen U $(x_2$ - Kill)

  - Definition 2:
    - $f(x_1 \wedge x_2) = (Gen \cup ((x_1 \cup x_2) - Kill))$

      $f(x_1) \wedge f(x_2) = (Gen \cup (x_1 - Kill)) \cup (Gen \cup (x_2 - Kill))$

Distributivity

- **A framework (F, V, ∧) is distributive if and only if**

  $f(x \wedge y) = f(x) \wedge f(y)$,

  meet input, then apply f is **equal to**
  apply the transfer function individually then merge result

Important Note

- Monotone framework **does not mean** that $f(x) \leq x$
  - e.g. Reaching definition for two definitions in program
  - suppose: f: Gen = $\{d_1\}$; Kill = $\{d_2\}$

III. Properties of Iterative Algorithm

- **Given:**

  ∧ and monotone data flow framework

  Finite descending chain

  ⇒ Converges

- **Initialization of interior points to T**

  ⇒ Maximum Fixed Point (MFP) solution of equations

Behavior of iterative algorithm (intuitive)

For each IN/OUT of an interior program point:

- Invariant: new value ≤ old value in any step
- Start with T (largest value)
- Proof by induction
  - 1st transfer function or meet operator: new value ≤ old value (T)
  - Meet operation:
    - Assume new inputs ≤ old inputs, new output ≤ old output
  - Transfer function (in a monotone framework)
    - Assume new inputs ≤ old inputs, new output ≤ old output
- Algorithm iterates until equations are satisfied
- Values do not come down unless some constraints drive them down.
- Therefore, finds the largest solution that satisfies the equations

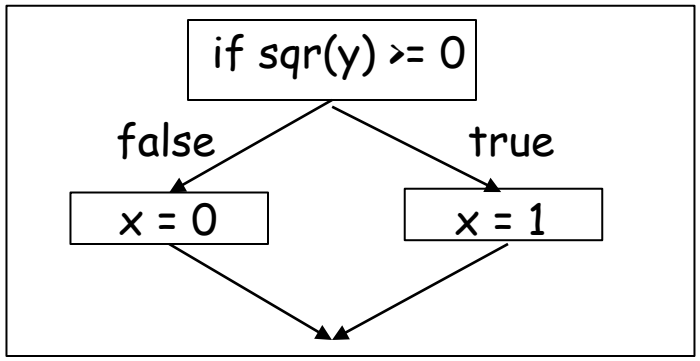
IV. What Does the Solution Mean?

- **IDEAL data flow solution**
  - Let $f_1, \ldots, f_m : \in F$, f_i is the transfer function for node i

    $f_p = f_{n_k} \bullet \ldots \bullet f_{n_1}$, p is a path through nodes $n_1, \ldots, n_k$

    f_p = identify function, if p is an empty path

  - For each node n: $\wedge f_{pi}$ (boundary value), for <u>all possibly executed paths</u> p_i reaching n
  - Example



- **Determining all possibly executed paths is undecidable**

Meet-Over-Paths MOP

- **Err in the conservative direction**

- **Meet-Over-Paths MOP**
  - Assume every edge is traversed
  - For each node n:

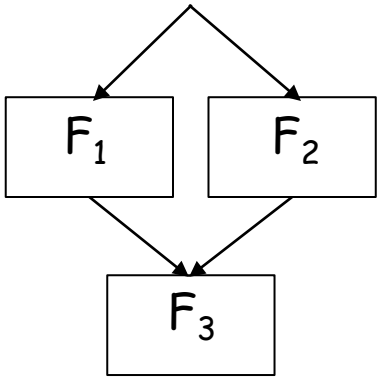
  - MOP(n) = $\wedge f_{p_i}$ (boundary value), for all paths p_i reaching n

- **Compare MOP with IDEAL**
  - MOP includes more paths than IDEAL
  - MOP = IDEAL $\wedge$ Result(Unexecuted-Paths)
  - MOP $\leq$ IDEAL
  - MOP is a "smaller" solution, more conservative, **safe**

- **MOP $\leq$ IDEAL**
  - Goal: as close to MOP from below as possible

Solving Data Flow Equations

- **What is the difference between MOP and MFP of data flow equations?**



- **Therefore**
  - FP ≤ MFP ≤ MOP ≤ IDEAL
  - FP, MFP, MOP are safe
  - If framework is distributive, FP ≤ MFP = MOP ≤ IDEAL

Summary

- **A data flow framework**
  - Semi-lattice
    - set of values (top)
    - meet operator
    - finite descending chains?
  - Transfer functions
    - summarizes each basic block
    - boundary conditions
- **Properties of data flow framework:**
  - Monotone framework and finite descending chains

    ⇒ iterative algorithm converges
    ⇒ finds maximum fixed point (MFP)
    ⇒ FP ≤ MFP ≤ MOP ≤ IDEAL

  - Distributive framework

    ⇒ FP ≤ MFP = MOP ≤ IDEAL