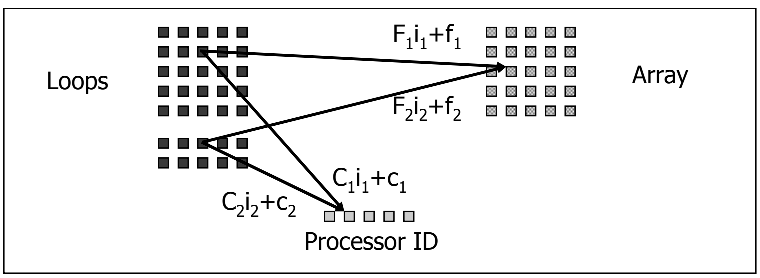
CS 243
Lecture 13
Affine Transforms

1. Loop Permutation as an Example
2. Seven Primitive Transforms
3. Advanced Topic: Pipelining & Blocking

Readings: Chapter 11.1 - 11.7

Affine Partitioning



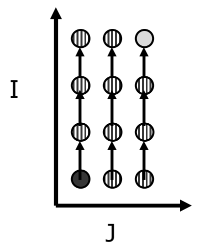
For every pair of data dependent accesses $F_1i_1+f_1$ and $F_2i_2+f_2$
Find C_1, c_1, C_2, c_2:

$$\forall\, i_1, i_2 \quad F_1 i_1 + f_1 = F_2 i_2 + f_2 \rightarrow C_1 i_1 + c_1 = C_2 i_2 + c_2$$

with the objective of maximizing the rank of C_1, C_2

1. Example:
Loop Interchange (Loop Permutation)

```
for I = 1 to 4
  for J = 1 to 3
    Z[I,J] = Z[I-1,J]
```



Data dependent operations:

$\forall$ i, j, i', j' such that,

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}i\\j\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}i'\\j'\end{bmatrix}+\begin{bmatrix}-1\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}i'-i\\j'-j\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \qquad \begin{bmatrix}i'-i\\j'-j\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

Find C_1, C_2, c for statement in loop

$$\begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}i\\j\end{bmatrix}+c=\begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}i'\\j'\end{bmatrix}+c$$

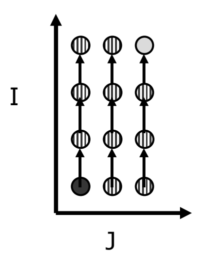
$$\begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}i'-i\\j'-j\end{bmatrix}=0 \qquad \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=0$$

One solution: $\begin{bmatrix}C_1 & C_2\end{bmatrix}=\begin{bmatrix}0 & 1\end{bmatrix}$ $\quad c = 0$

Therefore: $p=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}i\\j\end{bmatrix}+c \qquad p = j$

Code Generation: Execute each partition in sequential order
Must be correct!

```
for I = 1 to 4
  for J = 1 to 3
    Z[I,J] = Z[I-1,J]
```



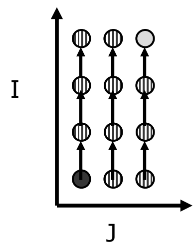

```
for P = 1 to 3
  for I' = 1 to 4
    for J' = 1 to 3
      if (J' = P)
        Z[I',J'] = Z[I'-1,J']
```

```
for P = 1 to 3
  for I' = 1 to 4
    Z[I',P] = Z[I'-1,P]
```

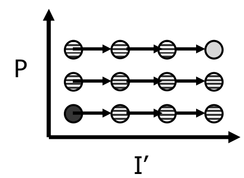
$$\begin{bmatrix}p\\i'\end{bmatrix}=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}i\\j\end{bmatrix}$$

Geometric Interpretation

```
for I = 1 to 4
  for J = 1 to 3
    Z[I,J] = Z[I-1,J]
```




```
for P = 1 to 3
  for I' = 1 to 4
    Z[I',P] = Z[I'-1,P]
```



$$\begin{bmatrix} p \\ i' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}$$

Affine Partitioning Algorithm: Maximize Degree of Parallelism with No Communication

- Find data dependences
- For each pair of data dependent operations
  - Set up equations $F_1 i_1 + f_1 = F_2 i_2 + f_2$ to capture relations of dependent iterations
    - Reduce the number of unknowns – lots of identities.
  - Set up equations for affine partitions $C_1 i_1 + c_1 = C_2 i_2 + c_2$
    - Each operation gets its own C, c
    - Work on C by dropping the c',
      Rewrite constraints as Ax = 0, where x is the unknown Cs
    - Nullity of A is the rank C
    - Find the basis vectors
    - Find the constant c

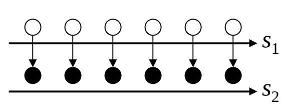
2. Primitive Loop Transformations

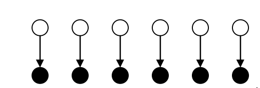
- Key idea:
  If you draw the dependence graph,
  you can eyeball the code and get the solution
  derived using linear algebra

- Seven source-level transformations
  - Unimodular transform: Reversal, permutation, skewing
  - Fusion, fission, re-indexing, scaling

- Affine partitioning can express arbitrary combinations of
  these seven primitives.

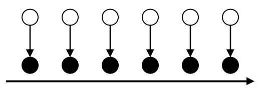
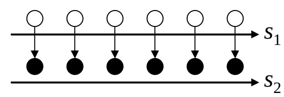
Fusion

<table>
<tr><th>Source Code</th><th>Partition</th><th>Transformed Code</th></tr>
<tr><td><pre>
for (i=1; i<=N; i++)
  Y[i] = Z[i]; /*s1*/
for (j=1; j<=N; j++)
  X[j] = Y[j]; /*s2*/
</pre>


</td><td>$s_1 : p = i$
$s_2 : p = j$</td><td><pre>
for (p=1; p<=N; p++) {
  Y[p] = Z[p];
  X[p] = Y[p];
}
</pre>

</td></tr>
</table>

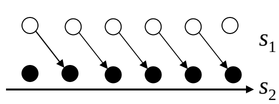
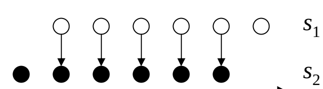
Fission

Source Code	Partition	Transformed Code

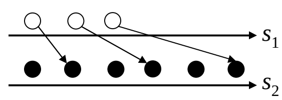
`for (p=1; p<=N; p++) {`
`  Y[p] = Z[p];`
`  X[p] = Y[p];`
`}`

	$s_1 : i = p$
	$s_2 : j = p$
		`for (i=1; i<=N; i++)`
		`  Y[i] = Z[i]; /*s1*/`
		`for (j=1; j<=N; j++)`
		`  X[j] = Y[j]; /*s2*/`
		

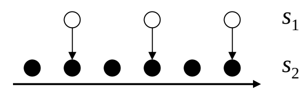
Re-indexing

Source Code	Partition	Transformed Code

`for (i=1; i<=N; i++) {`
`  Y[i] = Z[i];   /*s1*/`
`  X[i] = Y[i-1]; /*s2*/`
`}`

	$s_1 : p = i$
	$s_2 : p = i - 1$
		`if (N>=1) X[1]=Y[0];`
		`for (p=1; p<=N; p++) {`
		`  Y[p] = Z[p];`
		`  X[p+1] = Y[p];`
		`}`
		`if (N>=1) Y[N]=Z[N];`
		

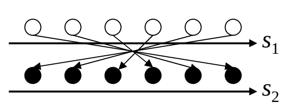
Scaling

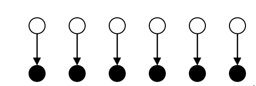
<table>
<tr><th>Source Code</th><th>Partition</th><th>Transformed Code</th></tr>
<tr><td><pre>
for (i=1; i<=N; i++)
  Y[2*i] = Z[2*i]; /*s1*/
for (j=1; j<=2*N; j++)
  X[j] = Y[j];   /*s2*/
</pre>


</td><td>$s_1 : p = 2 \times i$
$s_2 : p = j$</td><td><pre>
for (p=1; p<=2*N; p++){
  if (p mod 2 == 0)
    Y[p] = Z[p];
  X[p] = Y[p];
}
if (N>=1) Y[N]=Z[N];
</pre>


</td></tr>
</table>

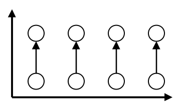
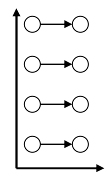
Reversal

<table>
<tr><th>Source Code</th><th>Partition</th><th>Transformed Code</th></tr>
<tr><td><pre>
for (i=0; i<=N; i++)
  Y[N-i] = Z[i]; /*s1*/
for (j=0; j<=N; j++)
  X[j] = Y[j];   /*s2*/
</pre>


</td><td>$s_1 : p = i$
$s_2 : p = j$</td><td><pre>
for (p=0; p<=N; p++) {
  Y[p] = Z[N-p];
  X[p] = Y[p];
}
</pre>

</td></tr>
</table>

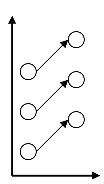
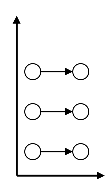
Permutation

Source Code	Partition	Transformed Code

`for (i=1; i<=N; i++)`
`  for (j=0; j<=M; j++)`
`     Z[i,j] = Z[i-1,j];`

	$\begin{bmatrix}p\\q\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}i\\j\end{bmatrix}$	`for (p=0; p<=M; p++)`
		`  for (q=1; q<=N; q++)`
		`     Z[q,p] = Z[q-1,p];`
		

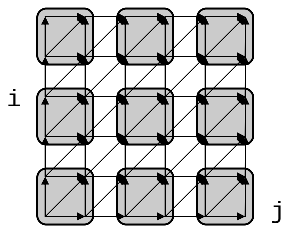
Skewing

Source Code	Partition	Transformed Code

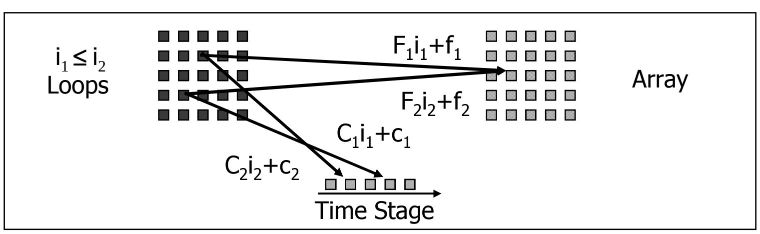
`for (i=1; i<=N+M-1; i++)`
`  for (j=max(1,i+N);`
`      j<=min(i,M); j++)`
`     Z[i,j] = Z[i-1,j-1];`

	$\begin{bmatrix}p\\q\end{bmatrix}=\begin{bmatrix}1&-1\\0&1\end{bmatrix}\begin{bmatrix}i\\j\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$	`for (p=1; p<=N; p++)`
		`  for (q=1; q<=M; q++)`
		`     Z[p,q-p] =`
		`       Z[p-1,q-p-1];`
		

3. Advanced topic: Pipelining
SOR (Successive Over-Relaxation): An Example

```
for i = 0 TO m
  for j = 0 to n
    X[j+1] = 1/3 * (X[j] + X[j+1] + X[j+2])
```



Finding the Maximum Degree of Pipelining



For every pair of data dependent accesses $F_1 i_1 + f_1$ and $F_2 i_2 + f_2$

Let $B_1 i_1 + b_1 \geq 0$, $B_2 i_2 + b_2 \geq 0$ be the corresponding loop bound constraints,

Find C_1, c_1, C_2, c_2:

$\forall\, i_1, i_2 \quad B_1 i_1 + b_1 \geq 0,\ B_2 i_2 + b_2 \geq 0$

$(i_1 \leq i_2) \wedge (F_1 i_1 + f_1 = F_2 i_2 + f_2) \rightarrow C_1 i_1 + c_1 \leq C_2 i_2 + c_2$

with the objective of maximizing the rank of C_1, C_2

Key Insight

- Choice in time mapping => (pipelined) parallelism
- Rank(C) – 1 degree of parallelism with 1 degree of synchronization
- Can create blocks with Rank(C) dimensions

- Find time partitions is not as straightforward as space partitions
  - Need to deal with linear inequalities
  - Solved using Farkas Lemma – no simple intuitive proof

Summary